Flattening stratification and the stack of partial stabilizations of prestable curves

Kresch, Andrew

Abstract: We describe the stack of partial stabilizations of prestable curves. We use this description to obtain examples of proper morphisms of schemes having no flattening stratification and to obtain results on the existence of global quotient stack presentations in the setting of moduli stack of prestable curves of genus 0

DOI: https://doi.org/10.1112/blms/bds066

Originally published at:
Kresch, Andrew (2013). Flattening stratification and the stack of partial stabilizations of prestable curves. Bulletin of the London Mathematical Society, 45(1):93-102.
DOI: https://doi.org/10.1112/blms/bds066

Bull. London Math. Soc. 45 (2013) 93–102
doi:10.1112/blms/bds066

Flattening stratification and the stack of partial stabilizations of prestable curves

Andrew Kresch

Abstract

We describe the stack of partial stabilizations of prestable curves. We use this description to obtain examples of proper morphisms of schemes having no flattening stratification and to obtain results on the existence of global quotient stack presentations in the setting of moduli stack of prestable curves of genus 0.

1. *Introduction*

Given a finite-type morphism of Noetherian schemes $f: T \to S$, a *flattening stratification* for f is a finite collection of pairwise disjoint locally closed subschemes $S_1, \dots, S_m$ of S, such that for any S-scheme S', the morphism $S' \times_S T \to S'$ obtained by base change is flat if and only if the structure morphism $S' \to S$ factors through $S_1 \amalg \dots \amalg S_m$. In [10], Grothendieck proved the existence of a flattening stratification when f is projective. If $\mathcal{O}(1)$ denotes a relatively very ample invertible sheaf on T, and $P_1(n), \dots, P_m(n)$ are the Hilbert polynomials of the fibres of f ordered by their values for $n \gg 0$, then a recipe to construct the S_i is as follows. If $n_0 \geqslant 0$ is chosen with $R^j f_*(\mathcal{O}(n)) = 0$ for all $n \geqslant n_0$ and $j \geqslant 1$, then, for suitable N, we can define open $U_i \subset S$ by the conditions $\dim(f_*(\mathcal{O}_T(n)) \otimes_{\mathcal{O}_S} \boldsymbol{k}(s)) \leqslant P_i(n)$ for $n_0 \leqslant n \leqslant N$, where $\boldsymbol{k}(s)$ denotes the residue field of $s \in S$, and the sum of Fitting ideals corresponding to conditions $\dim(f_*(\mathcal{O}_T(n)) \otimes_{\mathcal{O}_S} \boldsymbol{k}(s)) \geqslant P_i(n)$ for all $n \geqslant n_0$ defines $S_i \subset U_i$. For details, see [15].

When f is proper, Grothendieck has shown the existence of a scheme with a finite-type monomorphism to S, which takes the place of $S_1 \amalg \dots \amalg S_m$ in the above condition. The proof (see Murre's exposition [16]) uses a general representability criterion for unramified functors. Following Raynaud and Gruson [19], we call this S-scheme a *universal flatificator* for f. The question whether this is necessarily a stratification was raised by Olsson and Starr [18].

The purpose of this note is to use partial stabilizations of prestable curves to obtain examples where the universal flatificator is not a stratification. These examples are derived from an example worked out using the formalism of Hom-stacks of Olsson [17]. An étale cover of the moduli stack of prestable curves of genus 0 is exhibited, whose connected components are global quotient stacks, while as predicted by Edidin and Fulghesu [6] no global quotient presentation exists for sufficiently large finite-type open substacks of the moduli stack itself.

2. *Generalities*

2.1. *Conventions*

All schemes are quasi-separated. For algebraic spaces and algebraic stacks, we follow the conventions of Knutson [13] and Laumon and Moret-Bailly [14], respectively; in particular,

Received 14 March 2011; revised 22 February 2012; published online 11 September 2012.

2010 *Mathematics Subject Classification* 14H10 (primary), 14A15, 14D23 (secondary).

The author's research has been supported by a grant from the SNF.

the diagonal of every algebraic space or stack is separated and is of finite type. But, for an algebraic stack X, an algebraic stack U with representable smooth surjective morphism to X will be called an atlas for X. (In many sources, this terminology is reserved for a scheme U with smooth surjective morphism to X.)

We use the notation $\bar{M}_{g,n}$ for the moduli stack of stable n-pointed curves of genus g (see [**5**, **12**]). The moduli stack of prestable curves of genus g is denoted $\mathfrak{M}_g$ (see [**3**]). The universal curve over any moduli stack is denoted similarly, with the letter M replaced by C, for example, $\bar{C}_{g,n}$, $\mathfrak{C}_g$.

2.2. *Subsets of components*

If $f : X \to Y$ is a representable smooth finite-type morphism of algebraic stacks, then by Laumon and Moret-Bailly [**14**, (6.8)], there are an algebraic stack $\pi_0(X/Y)$ and a factorization of f as

$$X \longrightarrow \pi_0(X/Y) \longrightarrow Y,$$

where $\pi_0(X/Y) \to Y$ is representable étale, and $X \to \pi_0(X/Y)$ is smooth surjective with geometrically connected fibres.[†] We need the variant $\mathcal{P}\pi_0(X/Y)$: given any representable étale finite-type morphism $f : V \to Y$, we define $\mathcal{P}V$ to be the category of pairs consisting of a morphism $T \to Y$ and an open subspace $U \subset V \times_Y T$ such that $U \to T$ is universally closed ($\mathcal{P}V$ also may be described as $f_!\mathcal{F}$, where $\mathcal{F}$ is the constant sheaf of commutative monoids $\{\varnothing, \{*\}\}$ and $f_!$ is left adjoint to f^*). An object $(T \to Y, U \subset V \times_Y T)$ of $\mathcal{P}V$ will be called *maximal* when $U = V \times_Y T$.

Proposition 2.1. *For any representable étale finite-type morphism $f : V \to Y$ of algebraic stacks $\mathcal{P}V$ is an algebraic stack, the morphism $\mathcal{P}V \to Y$ is representable, étale and of finite type and there is an étale surjective morphism*

$$Y \amalg V \amalg (V \times_Y V) \amalg \ldots \longrightarrow \mathcal{P}V \tag{1}$$

mapping Y isomorphically to an open and closed substack of $\mathcal{P}V$.

The complement of the image of Y in $\mathcal{P}V$ will be denoted $\mathcal{P}V^{\text{non-}\varnothing}$.

Proof. Given a scheme T and morphism $T \to Y$, if U is open in $V \times_Y T$ and $U \to T$ is universally closed, then for any point of T there are an étale neighbourhood $T' \to T$ and a finite collection of sections of $V \times_Y T' \to T'$ such that $U \times_T T'$ is the union of their images. Thus, to show that $\mathcal{P}V \to Y$ is representable (which implies that $\mathcal{P}V$ is an algebraic stack) and étale and $\mathcal{P}V$ has the étale cover indicated in (1), it suffices to show that for any m and n the morphism

$$\Big(\bigtimes_Y^{(m)} V\Big) \times_{\mathcal{P}V} \Big(\bigtimes_Y^{(n)} V\Big) \longrightarrow \bigtimes_Y^{(m+n)} V$$

is representable and is an open immersion. This follows easily from the fact that $V \to Y$, being étale and representable, has relative diagonal which is an open immersion.

The last assertion holds, since Y and $\bigtimes_Y^{(n)} V$ have disjoint open images in $\mathcal{P}V$ for any $n \geqslant 1$. □

[†]The argument of Laumon and Moret-Bailly [**14**], which invokes [**11**, (15.6.4)], may be carried out when X is an algebraic space and Y is a scheme as soon as one recognizes that the local ring of an étale atlas may be used in place of the usual local ring in the proof of Grothendieck [**11**, (15.5.6)], and then $\pi_0(X/Y)$ as described here is a local construction [**14**, (14.1)]. For π_0 in greater generality, see [**20**].

2.3. *Prestable curves*

Recall that a family of prestable curves is a proper flat morphism $C \to T$ of finite presentation, where T is a scheme and C is an algebraic space, whose geometric fibres are connected, reduced and one-dimensional, and have only nodes as singularities. We denote the relative smooth locus by C^{sm}. Its complement C^{sing} is a closed algebraic subspace of C, defined on any affine étale chart by the Fitting ideal of the module of relative differentials.

When explicitly mentioned, we will also consider families of prestable curves where C and T are algebraic stacks and $C \to T$ is a representable morphism satisfying the above properties.

PROPOSITION 2.2. *Let $\psi : C \to T$ be a family of prestable curves. Then the morphisms $\pi_0(C^{\mathrm{sm}}/T) \to T$ and $\mathcal{P}\pi_0(C^{\mathrm{sm}}/T) \to T$ are universally closed.*

Proof. Using [**11**, (11.2.7)], we may reduce to the case T is Noetherian. Since $\pi_0(C^{\mathrm{sm}}/T) \to T$ is étale, universal closedness is equivalent to closedness upon base change to the strict henselization of $\mathcal{O}_{T,t}$ for all $t \in T$. So universal closedness is equivalent to closedness after base change to schemes of finite type over T.[†]

Now by standard arguments, it suffices to verify closedness after base change to Spec of a discrete valuation ring, so let us assume T to be of this form. Then it suffices to show that any point of $\pi_0(C^{\mathrm{sm}}/T)$ over the generic point of T has closure mapping surjectively to T. Equivalently, the generic point y of any irreducible component of C specializes to a smooth point in the fibre C_0 over the closed point of T. By properness, y specializes to some point of C_0. So y specializes to the generic point of some irreducible component of C_0.

The assertion for $\mathcal{P}\pi_0(C^{\mathrm{sm}}/T) \to T$ is now a consequence of Proposition 2.1. □

COROLLARY 2.3. *The conclusion of Proposition* 2.2 *is valid for a family of prestable curves in which C and T are algebraic stacks. Given sections s_1 and s_2 of $\pi_0(C^{\mathrm{sm}}/T) \to T$ and t_1 and t_2 of $\mathcal{P}\pi_0(C^{\mathrm{sm}}/T) \to T$, the substack of T defined by any of the following conditions is open*:

(i) *equality of s_1 and s_2*;
(ii) *equality of t_1 and t_2*;
(iii) *maximality of t_1*.

3. *Partial stabilization of prestable curves*

3.1. *Admissible sets of irreducible components*

Let g be a non-negative integer. For any n, there is a morphism $\bar{M}_{g,n} \to \mathfrak{M}_g$ which forgets the markings leaving the curve unchanged, and by Behrend [**3**, Proposition 2], the disjoint union of $\bar{M}_{g,n}$ over all n is an atlas for the Artin stack $\mathfrak{M}_g$. For a positive integer $n_1 \leqslant n$, there is a morphism

$$\bar{M}_{g,n} \longrightarrow \mathcal{P}\pi_0^{\text{non-}\varnothing}(\mathfrak{C}_g^{\mathrm{sm}}/\mathfrak{M}_g) \tag{2}$$

corresponding to the first n_1 sections. This makes the disjoint union of $\bar{M}_{g,n}$ over pairs of positive integers $n_1 \leqslant n$ an atlas for $\mathcal{P}\pi_0^{\text{non-}\varnothing}(\mathfrak{C}_g^{\mathrm{sm}}/\mathfrak{M}_g)$. The requirement that the component in the fibre of $\bar{C}_{g,n}^{\mathrm{sm}} \to \bar{M}_{g,n}$ of each of the first n_1 sections contains at least two others, among

[†] By Alper, Smyth and van der Wyck [**2**, Lemma 2.6], relying on [**22**, Tag 05BD], universal closedness for any quasi-compact morphism of algebraic stacks $X \to Y$ is equivalent to closedness after base change to schemes, finitely presented over Y.

the first n_1 sections, defines by Corollary 2.3 (applied to $\pi_0(\bar{C}^{\mathrm{sm}}_{g,n}/\bar{M}_{g,n}) \to \bar{M}_{g,n}$) an open substack $M_{g,n,n_1} \subset \bar{M}_{g,n}$. The disjoint union of M_{g,n,n_1} is again an atlas for $\mathcal{P}\pi_0^{\text{non-}\varnothing}(\mathfrak{C}^{\mathrm{sm}}_g/\mathfrak{M}_g)$.

We now consider the stabilization morphism [**12**] which forgets all but the first n_1 markings, performing contractions as necessary to enforce stability,

$$M_{g,n,n_1} \longrightarrow \bar{M}_{g,n_1}. \tag{3}$$

The substack of $\bar{M}_{g,n_1}$ where every component in the fibre of $\bar{C}^{\mathrm{sm}}_{g,n_1} \to \bar{M}_{g,n_1}$ has at least one marked point is open (by Corollary 2.3 applied to $\pi_0(\bar{C}^{\mathrm{sm}}_{g,n_1}/\bar{M}_{g,n_1}) \to \bar{M}_{g,n_1}$); we denote it by M'_{g,n_1}. We denote the pre-image of M'_{g,n_1} under the morphism (3) by M'_{g,n,n_1}.

Definition 1. The *admissible* locus in $\mathcal{P}\pi_0^{\text{non-}\varnothing}(\mathfrak{C}^{\mathrm{sm}}_g/\mathfrak{M}_g)$ is the union of the images of M'_{g,n,n_1} over pairs of positive integers $n_1 \leqslant n$, and will be denoted $\mathcal{P}'\pi_0(\mathfrak{C}^{\mathrm{sm}}_g/\mathfrak{M}_g)$.

3.2. *Partial stabilization locus in Hom-stack*

Stabilization induces a morphism of respective universal curves (cf. [**12**]), which in the present situation takes the form of a 2-commutative diagram

$$\begin{tikzcd} C'_{g,n,n_1} \arrow[r] \arrow[d] & C'_{g,n_1} \arrow[d] \\ M'_{g,n,n_1} \arrow[r, "\mathrm{st}'"'] & M'_{g,n_1} \end{tikzcd} \tag{4}$$

Proposition 3.1. *The universal curves* $\mathfrak{C}_g \times_{\mathfrak{M}_g} \mathcal{P}'\pi_0(\mathfrak{C}^{\mathrm{sm}}_g/\mathfrak{M}_g) \to \mathcal{P}'\pi_0(\mathfrak{C}^{\mathrm{sm}}_g/\mathfrak{M}_g)$ *and* $\mathfrak{C}_g \to \mathfrak{M}_g$ *fit into a 2-commutative diagram*

$$\begin{tikzcd} \mathfrak{C}_g \times_{\mathfrak{M}_g} \mathcal{P}'\pi_0(\mathfrak{C}^{\mathrm{sm}}_g/\mathfrak{M}_g) \arrow[r] \arrow[d] & \mathfrak{C}_g \arrow[d] \\ \mathcal{P}'\pi_0(\mathfrak{C}^{\mathrm{sm}}_g/\mathfrak{M}_g) \arrow[r, "\mathrm{st}"'] & \mathfrak{M}_g \end{tikzcd} \tag{5}$$

with horizontal arrows determined uniquely up to 2-isomorphism, such that the cube formed by (4) *and* (5) *is 2-commutative.*

Proof. There is a prestack, consisting of the objects of the stack $\coprod M'_{g,n,n_1}$ (disjoint union over pairs of positive integers $n_1 \leqslant n$) and the morphisms of families of curves preserving the sets of components picked out by the first n_1 sections, having stackification $\mathcal{P}'\pi_0(\mathfrak{C}^{\mathrm{sm}}_g/\mathfrak{M}_g)$. Arguing as in [**4**, p. 26] (proof of Claim 5 and the universality statement immediately following), we have a morphism from the prestack to $\mathfrak{M}_g$, fitting into a strictly commutative diagram with the morphism st' of (4). Over the prestack there is a universal curve, and from this there is a morphism to $\mathfrak{C}_g$, fitting into a 2-commutative cube with strictly commuting left, right, top and bottom faces. The proposition now follows by applying the universal property of stackification (twice in the form of existence of morphisms, unique up to canonical 2-isomorphism and once in the form of existence and uniqueness of a 2-isomorphism). $\square$

Theorem 3.2. *Partial stabilization in* (5) *determines a morphism*

$$\mathcal{P}'\pi_0(\mathfrak{C}^{\mathrm{sm}}_g/\mathfrak{M}_g) \longrightarrow \mathrm{Hom}_{\mathfrak{M}_g\times\mathfrak{M}_g}(\mathfrak{C}_g \times \mathfrak{M}_g, \mathfrak{M}_g \times \mathfrak{C}_g)$$

to the stack of morphisms [**17**] *of universal curves, which is an isomorphism onto an open and closed substack of* $\mathrm{Hom}_{\mathfrak{M}_g\times\mathfrak{M}_g}(\mathfrak{C}_g \times \mathfrak{M}_g, \mathfrak{M}_g \times \mathfrak{C}_g)$.

The proof makes use of the following preliminary result.

LEMMA 3.3. *Let k be a field, and let $f : C \to C'$ be a morphism of prestable curves over k. Then the following are equivalent.*

(i) *Over each irreducible component of C', the restriction of f has generic degree* 1, *and every geometric fibre of f is connected and has arithmetic genus* 0.

(ii) *We have $f_*\mathcal{O}_C \cong \mathcal{O}_{C'}$ and $R^1 f_*\mathcal{O}_C = 0$.*

(iii) *The morphism f is* (*up to isomorphism*) *the partial stabilization corresponding to some admissible set of irreducible components of C.*

Proof. The equivalence of (i) and (ii) follows from standard facts on sheaf cohomology (Stein Factorization, Theorem on Formal Functions), and the implication (iii) ⇒ (ii) follows from [**12**, Corollary 1.5]. Suppose (ii) holds. Now, it suffices to show that the set of irreducible components of C, corresponding to the irreducible components of C', is admissible, for then the isomorphism $f_*\mathcal{O}_C \cong \mathcal{O}_{C'}$ gives rise to an isomorphism of C' with the corresponding partial stabilization of C, compatible with f.

For the verification of admissibility, we may suppose k to be algebraically closed. The verification is trivial when f is an isomorphism, so we suppose the contrary, and we may by an inductive argument suppose that admissibility is known for any morphism satisfying (ii) where the source curve has fewer components than C has. We claim that C, endowed with three markings per selected component, is not stable. Indeed, for any $p' \in C'$ with $\dim f^{-1}(p') = 1$, we have $\omega_C|_{f^{-1}(p')}$ isomorphic to $\omega_{f^{-1}(p')}(p)$ when p' is a smooth point of C', or to $\omega_{f^{-1}(p')}(p_1 + p_2)$ when p' is a node, for some point p, respectively, points p_1 and p_2, on $f^{-1}(p')$. In either case, this is an invertible sheaf of non-positive degree. Let n_2 additional markings on the contracted components of C be chosen so that C with $n = n_1 + n_2$ markings (where n_1 is three times the number of irreducible components of C') is stable, and suppose that removal of the nth marking renders C unstable; let $f'' : C \to C''$ be the stabilization. Using $f''_*\mathcal{O}_C \cong \mathcal{O}_{C''}$, we obtain a factorization of f as the composite

$$C \xrightarrow{f''} C'' \xrightarrow{f'} C'$$

for some f', satisfying $f'_*\mathcal{O}_{C''} \cong \mathcal{O}_{C'}$ and $R^1 f'_*\mathcal{O}_{C''} = 0$. We conclude by applying the induction hypothesis. □

Proof of Theorem 3.2. The morphism is induced by diagram (5) and is representable by Laumon and Moret-Bailly [**14**, (3.12)]. We first show that the morphism is an open immersion. By Grothendieck [**11**, (17.9.1)], this is equivalent to saying that it is étale and injective on geometric points. The morphism is clearly injective on geometric points. To check that it is étale, we use the characterization of Grothendieck [**11**, (17.14.2)]. This says that it is enough to verify, for a local Noetherian ring A and square-zero ideal $I \subset A$, corresponding to closed $S_1 \subset S = \mathrm{Spec}(A)$, that for any morphism $f : C \to C'$ of prestable curves of genus g over S, whose restriction $f_1 : C_1 \to C'_1$ over S_1 is identified with a partial stabilization, there exists a unique extension to an identification of C' with a partial stabilization of C over S, compatible with f. By Proposition 2.1, the admissible section of $\mathcal{P}\pi_0(C^{\mathrm{sm}}/S)$ over S_1 extends uniquely to a section over S, which is again admissible, since the admissible locus is open. Let $C'' \to S$ be the corresponding family of prestable curves of genus g, with partial stabilization $C \to C''$ over S. By hypothesis, we have an isomorphism $C'_1 \to C''_1$ over S_1, fitting into a commutative diagram with C_1. Applying [**12**, Corollary 1.5], we see that $\mathcal{O}_{C'} \to f_*\mathcal{O}_C$ is an isomorphism (it is surjective by Nakayama's lemma, since after base change to the residue field it is surjective, and then injectivity is implied by injectivity after base change to the residue field). There is, similarly, an isomorphism with $\mathcal{O}_{C''}$. So, there is a unique compatible isomorphism $C' \cong C''$.

Now we show that the image is stable under specialization, that is, the image is closed as well as open. It suffices to show, given a complete discrete valuation ring R with algebraically closed residue field and morphism of prestable curves of genus g

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr] & & C' \arrow[dl] \\ & S & \end{tikzcd}$$

over $S = \mathrm{Spec}(R)$ whose restriction to the generic point satisfies the conditions of Lemma 3.3, that the morphism f_0 obtained by base change to the closed point also satisfies the conditions of Lemma 3.3.

We let K denote the field of fractions of R, and k the residue field. Arguing with an affine covering of C, we see that $f_*\mathcal{O}_C$ is torsion-free and hence flat over S. Then, using that $\mathcal{O}_{C'}$ is integrally closed in $\mathcal{O}_{C'} \otimes_R K$, we have $f_*\mathcal{O}_C \cong \mathcal{O}_{C'}$.

Since f is a morphism of flat projective S-schemes of relative dimension 1, the formation of $R^1 f_*\mathcal{O}_C$ commutes with arbitrary base change. In particular, we have $(R^1 f_*\mathcal{O}_C) \otimes_R k \cong R^1(f_0)_*\mathcal{O}_{C_0}$, where C_0 denotes the fibre over the closed point. So $R^1 f_*\mathcal{O}_C$ is supported at finitely many closed points, and the morphism $\mathcal{O}_{C'_0} \to (f_0)_*\mathcal{O}_{C_0}$ is an isomorphism away from these points.

Suppose p' lies in the support of $R^1 f_*\mathcal{O}_C$. Let E' be the union of irreducible components of C'_0 containing p', and let E be the union of irreducible components of C_0 mapping dominantly to a component of E'. If p' is a smooth point of C'_0, then there exists an affine neighbourhood V of p' in C', such that the pre-image U in C has the property that $U \cap E \to V \cap E'$ is an isomorphism, with some $p \in E$ mapping to p'. If p' is a node, then for a suitable neighbourhood there are points $p_1, p_2 \in E$ mapping to p' such that f induces an isomorphism $(U \cap E)/p_1 \sim p_2 \to V \cap E'$ from the curve with points p_1 and p_2 identified.

Now, we consider the exact sequence of sheaf cohomology groups

$$H^0(U, \mathcal{O}_U) \longrightarrow H^0(U_0, \mathcal{O}_{U_0}) \longrightarrow H^1(U, \mathcal{O}_U) \longrightarrow H^1(U, \mathcal{O}_U) \tag{6}$$

corresponding to multiplication by a uniformizing element of R. But $H^0(V_0, \mathcal{O}_{V_0})$ maps isomorphically to $H^0(U_0, \mathcal{O}_{U_0})$, from which it follows that the first map in (6) is surjective. This yields a contradiction. □

4. *No flattening stratification*

4.1. *Example with stacks*

We want to consider the universal partial stabilization $\mathfrak{C}_g \times_{\mathfrak{M}_g} \mathcal{P}'\pi_0(\mathfrak{C}_g^{\mathrm{sm}}/\mathfrak{M}_g) \to \mathcal{P}'\pi_0(\mathfrak{C}_g^{\mathrm{sm}}/\mathfrak{M}_g) \times_{\mathfrak{M}_g} \mathfrak{C}_g$, coming from the diagram (5). We do this in the case $g = 0$, and in order that we deal with finite-type algebraic stacks, we restrict to the locus where the curves under consideration have at most two nodes. This choice is made, since $m = 2$ is the smallest integer m such that the universal curve restricted to the locus with at most m nodes $\mathfrak{C}_0^{\leqslant m} \to \mathfrak{M}_0^{\leqslant m}$ is not projective [**6**].

Proposition 4.1. *The universal flatificator of*

$$\mathfrak{C}_0^{\leqslant 2} \times_{\mathfrak{M}_0^{\leqslant 2}} \mathcal{P}'\pi_0((\mathfrak{C}_0^{\leqslant 2})^{\mathrm{sm}}/\mathfrak{M}_0^{\leqslant 2}) \longrightarrow \mathcal{P}'\pi_0((\mathfrak{C}_0^{\leqslant 2})^{\mathrm{sm}}/\mathfrak{M}_0^{\leqslant 2}) \times_{\mathfrak{M}_0^{\leqslant 2}} \mathfrak{C}_0^{\leqslant 2} \tag{7}$$

is not a stratification.

Proof. Working over an arbitrary field k, we provide two morphisms

$$\mathbb{A}^1 \longrightarrow \mathcal{P}'\pi_0((\mathfrak{C}_0^{\leqslant 2})^{\mathrm{sm}}/\mathfrak{M}_0^{\leqslant 2}) \times_{\mathfrak{M}_0^{\leqslant 2}} \mathfrak{C}_0^{\leqslant 2}$$

that are identical as maps of the underlying sets of points [**14**, (5.2)], such that the morphism obtained by base change of (7) along one of them is flat and along the other is not flat. Since $\mathbb{A}^1$ is reduced, this is incompatible with the existence of a flattening stratification.

Using Theorem 3.2, it is enough to specify a source family, a target family, two partial stabilizations and a section:

$$\begin{tikzcd} C \arrow[rr, shift left, "f_1"] \arrow[rr, shift right, "f_2"'] \arrow[dr] & & C' \arrow[dl] \\ & \mathbb{A}^1 \arrow[ur, bend right] & \end{tikzcd}$$

Let C_1 be a family of conics degenerating to a union of two lines at $t = 0$:

$$C_1 = \mathrm{Proj}((k[t])[X, Y, Z]/(XY - tZ^2)).$$

Let $C_2 = \mathbb{A}^1 \times \mathbb{P}^1$, choose a section of each family, let C be the union of C_1 and C_2 glued along the respective sections and let $C' = C_2$ with its section. We define maps as follows:

(i) f_1, collapsing C_1 to its section and mapping C_2 isomorphically to C';
(ii) f_2, collapsing C_2 to its section and mapping C_1 onto C' so that over $t = 0$ the irreducible component containing the section is collapsed.

The pullback of f_1 by the section is isomorphic to the structure map of C_1, which is flat. The pullback of f_2 by the section is the union of one irreducible component dominating $\mathbb{A}^1$ and another supported over $t = 0$, which is not flat. □

4.2. *Examples with schemes*

The two families with their partial stabilizations appearing in the proof of Proposition 4.1 may be combined into one, where the base is a union of two lines $\mathrm{Spec}(A)$, with $A = k[s, t]/(st)$. In coordinates, after base change by the section we will obtain

$$\mathrm{Proj}(A[X, Y, Z]/(sX, XY - tZ^2)) \longrightarrow \mathrm{Spec}(A).$$

A computation as outlined in Section 1 shows that the flattening stratification is

$$\mathbb{A}^1 \amalg (\mathbb{A}^1 \smallsetminus \{0\}) \longrightarrow \mathrm{Spec}(A)$$

corresponding to the loci given by $s = 0$ and $s \neq 0$, respectively.

Proposition 4.2. *Let S be the projective line over a field k with two k-points identified* (*that is, a nodal cubic curve with rational tangent directions at the node*). *Then there exists a morphism*

$$S \longrightarrow \mathcal{P}'\pi_0((\mathfrak{C}_0^{\leqslant 2})^{\mathrm{sm}}/\mathfrak{M}_0^{\leqslant 2}) \times_{\mathfrak{M}_0^{\leqslant 2}} \mathfrak{C}_0^{\leqslant 2} \tag{8}$$

such that the morphism obtained from (7) *by base change is a proper morphism of finite-type schemes over k, whose universal flatificator is $\mathbb{A}^1 \to S$.*

Proof. Starting with the family of curves over $\mathrm{Spec}(A)$ (with $A = k[s, t]/(st)$ as above) given in coordinates by

$$\mathrm{Proj}(A[X, Y, Z, W]/(YW - stXZ, ZW - sX^2, XY - tZ^2)), \tag{9}$$

we may consider the étale equivalence relation which identifies $(X : Y : Z : W)$ over $(0, t) \in \operatorname{Spec}(A)$, $t \neq 0$, with $(XW : XZ : t^{-1}X^2 : W^2) = (0 : Y : Z : 0)$ over $(t^{-1}, 0)$, and identifies $(X : Y : Z : W)$ over $(s, 0)$, $s \neq 0$, with $(s^{-1}Z^2 : Y^2 : YZ : XZ) = (X : 0 : 0 : W)$ over $(0, s^{-1})$. This is compatible with the morphism to $\mathbb{P}^1$ given by $(X : W) = (Z : sX) = (Y : 0)$. So, with the constant section $(1 : 0)$ of $S \times \mathbb{P}^1 \to S$, we obtain a diagram of algebraic spaces

$$\begin{tikzcd} C \arrow[rr] \arrow[dr] & & S \times \mathbb{P}^1 \arrow[dl] \\ & S \arrow[ur, bend right] & \end{tikzcd}$$

and hence, by Theorem 3.2, a morphism (8). But C is a scheme, as we may see by representing S as a quotient of a projective scheme by a free action of a group of order 2, so that C is such a quotient of a proper scheme that is covered by two open subschemes, each isomorphic to (9). Now it may be checked that any two points identified by the group action are contained in a common affine neighbourhood. □

This example occurs in the family given in [**8**, Example 2.3] of prestable curves over a projective algebraic surface such that the three-dimensional total space is not a scheme. Following [**8**], we have

Proposition 4.3. *Let k be a field. Then there exist the following.*

(i) *A morphism $C \to S \times \mathbb{P}^1$, the partial stabilization of a family of curves over a smooth projective surface S over k, with C a smooth algebraic space, such that the universal flatificator is not a stratification.*

(ii) *A finite étale morphism $\tilde{C} \to C$ such that $\tilde{C}$ is a scheme, and the universal flatificator of $\tilde{C} \to S \times \mathbb{P}^1$ is not a stratification.*

5. *Quotient stacks*

An algebraic stack X of finite type over a field is said to be a global quotient stack if X is isomorphic to a stack quotient $[Y/G]$ for some finite-type algebraic space Y with action of a linear algebraic group G (see [**7**]). Over a general Noetherian base scheme, one takes the same definition with G being a flat subgroup scheme of GL_n for some n; in fact, it is equivalent to require $G = \mathrm{GL}_n$. The (stronger) condition to be isomorphic to a quotient stack of the form $[Y/\mathrm{GL}_n]$ with Y a quasi-affine scheme is connected with the resolution property (of coherent sheaves, by locally free coherent sheaves); cf. [**9**, **21**, **23**].

5.1. *Étale covers by global quotient stacks*

An easy construction produces an étale cover of $\mathfrak{M}_0^{\leqslant m}$ for any m by a global quotient stack.

Proposition 5.1. *For any positive integer m, there exists a representable étale surjective morphism from a quotient stack of the form $[Y/\mathrm{GL}_n]$ with Y quasi-affine to the stack $\mathfrak{M}_0^{\leqslant m}$ of prestable curves of genus 0 with at most m nodes.*

Proof. A genus-0 prestable curve with at most m nodes has (geometrically) at most $m+1$ irreducible components. Consider now the morphism

$$\bigtimes^{(m+1)}_{\mathfrak{M}_0^{\leqslant m}} \pi_0((\mathfrak{C}_0^{\leqslant m})^{\mathrm{sm}}/\mathfrak{M}_0^{\leqslant m}) \longrightarrow \mathcal{P}\pi_0((\mathfrak{C}_0^{\leqslant m})^{\mathrm{sm}}/\mathfrak{M}_0^{\leqslant m}).$$

By Corollary 2.3, there is an open substack U of the stack on the left, where the $m+1$ selected components are all the components. The universal curve over $\pi_0((\mathfrak{C}_0^{\leqslant m})^{\mathrm{sm}}/\mathfrak{M}_0^{\leqslant m})$ has a natural line bundle, of degree 2 on the selected component and degree 0 on all other components. The tensor product of various powers of these line bundles on the universal cover over U has positive degree on every irreducible component of every geometric fibre. Now $U \to \mathfrak{M}_0^{\leqslant m}$ is representable, étale and surjective, and by standard techniques (cf. [**5**, Section 1]) U is isomorphic to the quotient stack of an open subscheme of a Hilbert scheme by the group of projective linear transformations, hence by Totaro [**23**] is isomorphic to a quotient stack of the form $[Y/\mathrm{GL}_n]$ with Y quasi-affine, for some n. $\square$

Remark 1. In the terminology of Rydh [**21**], the fact that $\mathfrak{M}_0^{\leqslant m}$ admits a representable étale surjective morphism from a quotient stack of the form $[Y/\mathrm{GL}_n]$ with Y quasi-affine is expressed by saying that $\mathfrak{M}_0^{\leqslant m}$ is of *global type*.

5.2. *No global quotient stack presentation*

In [**6**], it is shown that $\mathfrak{M}_0^{\leqslant 1}$ admits a global quotient stack presentation, and evidence is given for $\mathfrak{M}_0^{\leqslant m}$ not to be a global quotient stack for $m \geqslant 2$.

Proposition 5.2. *Over any base field, there is no global quotient stack presentation for the moduli stack $\mathfrak{M}_0^{\leqslant m}$ of genus-0 prestable curves with at most m nodes, for any $m \geqslant 2$.*

Proof. Let k denote the base field. There is a smooth surjective morphism $\varphi : \mathbb{A}^1 \to \mathfrak{M}_0^{\leqslant 1}$ corresponding to the family of curves C_1 from Section 4.1.

It suffices to treat the case $m = 2$. Now, $\mathcal{P}'\pi_0((\mathfrak{C}_0^{\leqslant 2})^{\mathrm{sm}}/\mathfrak{M}_0^{\leqslant 2})$ admits a global section over $\mathfrak{M}_0^{\leqslant 2}$ which is maximal over $\mathfrak{M}_0^{\leqslant 1}$ and selects the two 'tail' components over the locus of curves with two nodes. Indeed, this is a constructible locus in $\mathcal{P}'\pi_0((\mathfrak{C}_0^{\leqslant 2})^{\mathrm{sm}}/\mathfrak{M}_0^{\leqslant 2})$, readily verified to be stable under generization, and its projection to $\mathfrak{M}_0^{\leqslant 2}$ is étale and bijective on geometric points. So it defines a section. We observe that the composite $\mathfrak{M}_0^{\leqslant 2} \to \mathcal{P}'\pi_0((\mathfrak{C}_0^{\leqslant 2})^{\mathrm{sm}}/\mathfrak{M}_0^{\leqslant 2}) \to \mathfrak{M}_0$ of the restriction of the morphism st of (5) to $\mathcal{P}'\pi_0((\mathfrak{C}_0^{\leqslant 2})^{\mathrm{sm}}/\mathfrak{M}_0^{\leqslant 2})$ with this section has image $\mathfrak{M}_0^{\leqslant 1}$. Denoting the composite morphism to $\mathfrak{M}_0^{\leqslant 1}$ by σ, we consider the cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathbb{A}^1 \arrow[d, "\varphi"] \\ \mathfrak{M}_0^{\leqslant 2} \arrow[r, "\sigma"'] & \mathfrak{M}_0^{\leqslant 1} \end{tikzcd} \qquad (10)$$

The stack X has non-trivial stabilizer group at the point corresponding to a curve with two nodes, and X is smooth since the morphism φ is smooth. Now, it follows that X is not a global quotient stack either because X is irreducible and normal and has an open substack isomorphic to $\mathbb{A}^1$ with complement of codimension 2, hence possesses no non-trivial vector bundles, or because as may be observed directly the diagonal of X is not quasi-affine (cf. [**7**, Section 2], respectively, [**23**, Section 6]). So $\mathfrak{M}_0^{\leqslant 2}$ is not a global quotient stack. $\square$

Remark 2. Using cohomology and base change machinery as in [**12**], it emerges that the morphism σ in (10) acts by

$$(C \xrightarrow{\pi} T) \longmapsto \mathrm{Proj}\left(\bigoplus_{\ell \geqslant 0} \pi_*((\omega_{C/T}^{\vee})^{\otimes \ell})\right).$$

This observation permits a direct proof of Proposition 5.2, not using admissible sets of components or Hom-stacks.

REMARK 3. The stack X considered in the proof has $\mathbb{G}_m$ stabilizer at one point and possesses an étale cover by a quotient stack $[\mathbb{A}^2/\mathbb{G}_m]$ with action by weights 1 and -1. Thus, X satisfies [**1**, Conjecture 1], a statement on the étale-local structure of Artin stacks.

Acknowledgements. The author thanks Joseph Ayoub and David Rydh for helpful discussions. The author also thanks Johan de Jong for comments and questions that inspired some of these discussions in a July 2010 posting about flattening stratification on his blog.

References

1. J. ALPER, 'On the local quotient structure of Artin stacks', *J. Pure Appl. Algebra* 214 (2010) 1576–1591.
2. J. ALPER, D. SMYTH and F. VAN DER WYCK, 'Weakly proper moduli stacks of curves', Preprint, 2011, arXiv:1012.0538v2.
3. K. BEHREND, 'Gromov–Witten invariants in algebraic geometry', *Invent. Math.* 127 (1997) 601–617.
4. K. BEHREND and YU. MANIN, 'Stacks of stable maps and Gromov–Witten invariants', *Duke Math. J.* 85 (1996) 1–60.
5. P. DELIGNE and D. MUMFORD, 'The irreducibility of the space of curves of given genus', *Publ. Math. Inst. Hautes Études Sci.* 36 (1969) 75–109.
6. D. EDIDIN and D. FULGHESU, 'The integral Chow ring of the stack of at most 1-nodal rational curves', *Comm. Algebra* 36 (2008) 581–594.
7. D. EDIDIN, B. HASSETT, A. KRESCH and A. VISTOLI, 'Brauer groups and quotient stacks', *Amer. J. Math.* 123 (2001) 761–777.
8. D. FULGHESU, 'The stack of rational curves', *Comm. Algebra* 38 (2010) 2405–2417.
9. P. GROSS, 'Vector bundles as generators on schemes and stacks', PhD Thesis, Universität Düsseldorf, 2010.
10. A. GROTHENDIECK, 'Techniques de construction et théorèmes d'existence en géométrie algébrique IV: Les schémas de Hilbert', *Séminaire Bourbaki* 13 (1960/61) no. 221.
11. A. GROTHENDIECK, 'Éléments de géométrie algébrique, IV', *Inst. Hautes Études Sci. Publ. Math.* 20, 24, 28, 32 (1964–67).
12. F. F. KNUDSEN, 'The projectivity of the moduli space of stable curves, II: the stacks $M_{g,n}$', *Math. Scand.* 52 (1983) 161–199.
13. D. KNUTSON, *Algebraic spaces*, Lecture Notes in Mathematics 203 (Springer, Berlin, 1971).
14. G. LAUMON and L. MORET-BAILLY, *Champs algébriques* (Springer, Berlin, 2000).
15. D. MUMFORD, *Lectures on curves on an algebraic surface* (Princeton University Press, Princeton, NJ, 1966).
16. J. P. MURRE, 'Representation of unramified functors: applications (according to unpublished results of A. Grothendieck)', *Séminaire Bourbaki* 17 (1964/65) no. 294.
17. M. OLSSON, 'Hom-stacks and restriction of scalars', *Duke Math. J.* 134 (2006) 139–164.
18. M. OLSSON and J. STARR, 'Quot functors for Deligne–Mumford stacks', *Comm. Algebra* 31 (2003) 4069–4096.
19. M. RAYNAUD and L. GRUSON, 'Critères de platitude et de projectivité: technique de "platification" d'un module', *Invent. Math.* 13 (1971) 1–89.
20. M. ROMAGNY, 'Composantes connexes et irréductibles en familles', *Manuscripta Math.* 136 (2011) 1–32.
21. D. RYDH, 'Noetherian approximation of algebraic spaces and stacks', Preprint, 2011 http://www.math.kth.se/~dary/papers.html.
22. STACKS PROJECT AUTHORS, 'Stacks project', Web-based project, http://math.columbia.edu/algebraic_geometry/stacks-git/.
23. B. TOTARO, 'The resolution property for schemes and stacks', *J. reine angew. Math.* 577 (2004) 1–22.

Andrew Kresch
Institut für Mathematik
Universität Zürich
Winterthurerstrasse 190
CH-8057 Zürich
Switzerland

andrew.kresch@math.uzh.ch